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Supersymmetry, quark confinement and the harmonic oscillator

S A Bruce and P C Minning

Department of Physics, University of Concepcion, PO Box 160C, Concepcion, Chile

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Abstract
We study some quantum systems described by noncanonical commutation relations formally expressed as $[\widehat{q}, \widehat{p}] = \mathrm{i}\hbar(\widehat{I} + \widehat{\chi}\widehat{H}_{\mathrm{HO}})$, where $\widehat{H}_{\mathrm{HO}}$ is the associated (harmonic-oscillator-like) Hamiltonian of the system and $\widehat{\chi}$ is a Hermitian (constant) operator, i.e. $[\widehat{H}_{\mathrm{HO}}, \widehat{\chi}] = \widehat{0}$. In passing, we also consider a simple ($\widehat{\chi} = \widehat{0}$ canonical) model, in the framework of a relativistic Klein–Gordon-like wave equation.

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1. Introduction

This paper deals with some possible applications of certain harmonic-oscillator-like models to particle physics. From the beginning we should mention that the spirit of this paper is rather formal and analytical. Since the beginning of modern physics, an enormous amount of work has been produced regarding the harmonic oscillator: its various forms and applications to quantum and classical mechanics. It is clear to everyone that the concept of harmonic oscillator is a fundamental source to understand many concrete problems in physics [1]. Therefore, we content ourselves by presenting certain mathematical relations between deep-seated symmetries in which some specific kinds of harmonic oscillators are involved. Hence, it is not presently our goal to give explicit numerical solutions to the problems we address, since they are available (under various concepts) in the literature.

The main premise of this paper is to study some possible quantum systems with noncanonical commutation relations of the form

$$[\widehat{q}^i, \widehat{p}^j] = \mathrm{i}\hbar\delta^{ij}(\widehat{I} + \widehat{\chi}\widehat{H}_{\mathrm{HO}}), \tag{1}$$

where $\widehat{H}_{\mathrm{HO}}$ is the harmonic-oscillator-like Hamiltonian (whose specific form depends on the concrete nature of the treated system) and $\widehat{\chi}$ is a Hermitian (constant) operator, i.e. $[\widehat{H}_{\mathrm{HO}}, \widehat{\chi}] = \widehat{0}$.

In section 2, we present a particular type of harmonic oscillator in (1 + 1)-dimensions as an underlying device in its connection with supersymmetry and quark confinement. Sections 3 concerns some corresponding generalizations to (3 + 1)-dimensions.

2. A harmonic oscillator in (1 + 1)-dimensions

We begin by considering a quantum system consisting of a particle of mass m_0 moving in a (1 + 1)-dimensional spacetime described by the (noncanonical) 'coordinate and momentum' realizations:

$$\widehat{Q} \equiv \frac{c}{\omega}\sqrt{\frac{1}{2\hbar m_0\omega}}(m_0\omega\sigma_2\widehat{q} + \sigma_1\widehat{p}), \qquad \widehat{P} \equiv \frac{\hbar\omega}{c}\sqrt{\frac{1}{2\hbar m_0\omega}}(\sigma_2\widehat{p} - m_0\omega\sigma_1\widehat{q}), \tag{2}$$

where σ_j are Pauli matrices. In equation (2), the operators $\widehat{q}, \widehat{p}$ are the usual canonical coordinates, so that $[\widehat{q}, \widehat{p}] = \mathrm{i}\hbar\widehat{I}$. Here $\widehat{Q}$ and $\widehat{P}$ are Hermitian and formally traceless operators: $\widehat{Q}^\dagger = \widehat{Q}$, $\widehat{P}^\dagger = \widehat{P}$ and $\mathrm{Tr}(\widehat{Q}) = \mathrm{Tr}(\widehat{P}) = 0$. Next, we determine $\widehat{Q}\widehat{P}$ and $\widehat{P}\widehat{Q}$:

$$\widehat{Q}\widehat{P} = \frac{1}{2}(\widehat{q}\widehat{p} - \widehat{p}\widehat{q}) + \frac{1}{\omega}\sigma_1\sigma_2\left(\frac{\widehat{p}^2}{2m_0} + \frac{m_0\omega^2}{2}\widehat{q}^2\right) \equiv \frac{\mathrm{i}\hbar}{2}\widehat{I} + \frac{\mathrm{i}}{\omega}\sigma_3\widehat{H}_{\mathrm{HO}} = \frac{\mathrm{i}\sigma_3}{\omega}\widehat{H}_{\mathrm{SS}}, \tag{3}$$

$$\widehat{P}\widehat{Q} = -\left(\frac{\mathrm{i}\hbar}{2}\widehat{I} + \frac{\mathrm{i}}{\omega}\sigma_3\widehat{H}_{\mathrm{HO}}\right) = -\mathrm{i}\frac{\sigma_3}{\omega}\widehat{H}_{\mathrm{SS}}, \tag{4}$$

in which $\widehat{H}_{\mathrm{HO}} \equiv (1/2)(\widehat{p}^2/m_0 + m_0\omega^2\widehat{q}^2)$ and $\widehat{H}_{\mathrm{SS}}$ is the well-known supersymmetric Hamiltonian for the harmonic oscillator [2]. Thus,

$$[\widehat{Q}, \widehat{P}] = \mathrm{i}\hbar\left(\widehat{I} + \left(\frac{2\sigma_3}{\hbar\omega}\right)\widehat{H}_{\mathrm{HO}}\right) = -\frac{2\mathrm{i}\sigma_3}{\omega}\widehat{H}_{\mathrm{SS}}, \qquad \{\widehat{Q}, \widehat{P}\} = \widehat{0}, \tag{5}$$

where $\widehat{\chi} = 2\sigma_3/\hbar\omega$ and σ_3 may be interpreted as a 'charge operator'. Furthermore,

$$\begin{aligned} \widehat{Q}^2 &= \frac{c}{\hbar\omega^3}\left(\frac{\widehat{p}^2}{2m_0} + \frac{m_0\omega^2}{2}\widehat{q}^2 + \frac{\hbar\omega}{2}\sigma_3\right) = \frac{c}{\hbar}\left(\frac{1}{\omega}\right)^2\left(\frac{1}{\omega}\right)\widehat{H}_{\mathrm{SS}}, \\ \widehat{P}^2 &= \frac{\hbar\omega}{c^2}\left(\frac{\widehat{p}^2}{2m_0} + \frac{m_0\omega^2}{2}\widehat{q}^2 + \frac{\hbar\omega}{2}\sigma_3\right) = \frac{\hbar\omega}{c^2}\widehat{H}_{\mathrm{SS}}, \end{aligned} \tag{6}$$

from which we get

$$\widehat{H}_{\mathrm{SS}} \equiv \frac{1}{2}\hbar\omega\left\{\left(\frac{c}{\hbar\omega}\right)^2\widehat{P}^2 + \left(\frac{\omega}{c}\right)^2\widehat{Q}^2\right\} = \frac{\widehat{p}^2}{2m_0} + \frac{1}{2}m_0\omega^2\widehat{q}^2 + \frac{\hbar\omega}{2}\sigma_3. \tag{7}$$

Additionally, from equations (6) it follows that

$$[\widehat{H}_{\mathrm{SS}}, \widehat{Q}] = [\widehat{Q}^2, \widehat{Q}] = \widehat{0}, \qquad [\widehat{H}_{\mathrm{SS}}, \widehat{P}] = [\widehat{P}^2, \widehat{P}] = \widehat{0}, \tag{8}$$

which closes the algebra and confirms that the (positive-definite) Hamiltonian H_{SS} is invariant under this supersymmetry.

Finally,

$$[\widehat{H}_{\mathrm{HO}}, \widehat{Q}] = -\mathrm{i}\frac{c^2}{\omega}\widehat{P}, \qquad [\widehat{H}_{\mathrm{HO}}, \widehat{P}] = \mathrm{i}\omega\left(\frac{\hbar\omega}{c}\right)^2\widehat{Q}. \tag{9}$$

From equation (5), the operator $\widehat{H} \equiv (2\sigma_3/\hbar\omega)\widehat{H}_{\mathrm{HO}}$ incorporates negative-energy eigenvalues into the system. In fact, the structure of this equation reminds us of the Zitterbewegung (ZB) phenomenon (as a result of the eventual interference between positive- and negative-energy eigenstates) extensively studied by Barut [3] amongst others. On the other hand, note that, from the point of view of SUSYQM, $\widehat{Q}$ and $\widehat{P}$ are proportional to SUSY charges [4].

Thus, we have encountered three non-equivalent Hamiltonians: $\widehat{H}_{\mathrm{HO}}, \widehat{H}_{\mathrm{SS}}$ (boson–fermion symmetry) and $\widehat{H}$ (particle–antiparticle symmetry). In principle, each one of them refers to a different (in nature) quantum system.

2.1. *Modeling 'quark confinement' in* $(1+1)$*-dimensions*

In $(1+1)$-dimensions, some authors [5–11] have argued that a modification of the usual canonical coordinates $\widehat{q}, \widehat{p}$ into a set of noncanonical coordinates (say) $\widehat{Q}, \widehat{P}$, could lead to a modification of the usual commutation relation, similar to the first equation (5), into the form

$$[\widehat{Q}, \widehat{P}] = \mathrm{i}\hbar(\widehat{I} + \chi \widehat{H}), \tag{10}$$

where χ is just a parameter with dimensions of $[\text{Energy}]^{-1}$. With this (oversimplified) scheme it is possible to attempt to give some hints in order to describe new phenomena, which appear, for instance, in high energy physics and mesoscopic systems. In [5], it is shown that for a quantum system fulfilling commutation relation (10) space discreteness is compatible with Lorentz transformations. This fact was explicitly related to atomic phenomena. In [6, 7], a mass spectrum for a subset of elementary particles was obtained from equation (10), with $\widehat{H}$ being the Hamiltonian of the harmonic oscillator and applied for energies of the order of GeV–TeV (10^{10}–10^{12} eV). In [8] it was also shown that, for the free-particle Hamiltonian, equation (10) reproduces space quantization. This result can be related to quark confinement phenomena. Qualitatively, in equation (10) as $\chi \to \infty$, the theory becomes 'asymptotically free' since $\widehat{H} \propto \widehat{\mathbf{p}}^2$. On the other hand, if $\chi \to 0$, $\widehat{H} \propto \widehat{\mathbf{q}}^2$, the parameter $1/\chi$ corresponds to a strong coupling constant: 'quark confinement in $(1+1)$-dimensions'. Furthermore, in [9, 10] mathematical aspects of equation (10) were studied. In [11], it was found that charge discreteness in mesoscopic circuits can be mathematically formulated with commutation relations similar to that of equation (10) between charge and current. This theory becomes related to the descriptions of phenomena like persistent current in a ring of inductance L, i.e., Coulomb blockage phenomena in a pure capacitor design.

3. Some generalizations in (3 + 1)-dimensions

At least three straightforward (non-equivalent) quantum models can be visualized in $(3+1)$-dimensions.

3.1. *Supersymmetry algebra*

By somehow mimicking the $(1+1)$-dimensional case, we address a quantum system consisting of a spin-$1/2$ particle of mass m_0 moving in a $(3+1)$-dimensional spacetime described by the noncanonical 'coordinate and momentum' Hermitian operators

$$\widehat{Q} \equiv \frac{c}{\omega}\sqrt{\frac{1}{6\hbar m_0\omega}}(m_0\omega\alpha \cdot \widehat{\mathbf{q}} + \mathrm{i}\beta\alpha \cdot \widehat{\mathbf{p}}), \qquad \widehat{P} \equiv \frac{\hbar\omega}{c}\sqrt{\frac{1}{6\hbar m_0\omega}}(\alpha \cdot \widehat{\mathbf{p}} - m_0\omega\, \mathrm{i}\beta\alpha \cdot \widehat{\mathbf{q}}), \tag{11}$$

where $\beta, \alpha_j (j = 1, 2, 3)$ are Dirac matrices. The operators $\widehat{q}^i, \widehat{p}^j$ are the usual canonical coordinates: $[\widehat{q}^i, \widehat{p}^j] = \mathrm{i}\hbar\widehat{I}\delta^{ij}$. Here $\widehat{\mathbf{Q}}$ and $\widehat{\mathbf{P}}$ are formally traceless operators: $\mathrm{Tr}(\widehat{\mathbf{Q}}) = \mathrm{Tr}(\widehat{\mathbf{P}}) = 0$. Next, we determine $\widehat{\mathbf{Q}}\,\widehat{\mathbf{P}}$ and $\widehat{\mathbf{P}}\,\widehat{\mathbf{Q}}$:

$$\widehat{\mathbf{Q}}\,\widehat{\mathbf{P}} = \frac{1}{2} \times \frac{1}{3}(\widehat{\mathbf{q}} \cdot \widehat{\mathbf{p}} - \widehat{\mathbf{p}} \cdot \widehat{\mathbf{q}}) + \frac{\mathrm{i}}{3\omega}\beta \times \left\{\frac{\widehat{\mathbf{p}}^2}{2m_0} + \frac{m_0\omega^2}{2}\widehat{\mathbf{q}}^2 + \frac{2\omega}{\hbar}\beta\mathbf{S} \cdot \mathbf{L}\right\} \equiv \frac{\mathrm{i}\hbar}{2}\widehat{I} + \left(\frac{\mathrm{i}\beta}{3\omega}\right)\widehat{H}_{\mathrm{SS}}, \tag{12}$$

where now

$$\widehat{H}_{\mathrm{SS}} \equiv \widehat{H}_{\mathrm{HO}} + \frac{2\omega}{\hbar}\beta\mathbf{S} \cdot \mathbf{L} = \frac{\widehat{\mathbf{p}}^2}{2m_0} + \frac{m_0\omega^2}{2}\widehat{\mathbf{q}}^2 + \frac{2\omega}{\hbar}\beta\mathbf{S} \cdot \mathbf{L} \tag{13}$$

is the Dirac oscillator [12–17], except for the additive term $3m_0c^2$, in which $\widehat{H}_{\rm SS}$ picks up a spin–orbit term, with $\mathbf{L} = \widehat{\mathbf{q}}\times\widehat{\mathbf{p}}$ the orbital angular momentum of m_0 and $S_k \equiv (1/2)\,\Sigma_k = (1/2)\,\alpha_i\alpha_j$ is the spin of m_0, with i, j, k cyclically. Correspondingly,

$$\widehat{\mathbf{P}}\widehat{\mathbf{Q}} = -\frac{\mathrm{i}\hbar}{2}\widehat{I} - \left(\frac{\mathrm{i}\beta}{3\omega}\right)\widehat{H}_{\rm SS}. \tag{14}$$

Therefore, we now get

$$[\widehat{\mathbf{Q}}, \widehat{\mathbf{P}}] = \mathrm{i}\hbar\left(\widehat{I} + \left(\frac{2}{3}\frac{\beta}{\hbar\omega}\right)\widehat{H}_{\rm SS}\right), \qquad \{\widehat{\mathbf{Q}}, \widehat{\mathbf{P}}\} = \widehat{0}, \tag{15}$$

together with

$$[\widehat{H}_{\rm HO}, \widehat{Q}^j] = -\mathrm{i}\frac{c^2}{\omega}\widehat{P}^j, \qquad [\widehat{H}_{\rm HO}, \widehat{P}^j] = \mathrm{i}\omega\left(\frac{\hbar\omega}{c}\right)^2\widehat{Q}^j,$$

in which $\widehat{\chi} = 2\beta/3\hbar\omega$ and β may be interpreted as a 'charge operator'. From its definition, $\widehat{H} \equiv -(3/2)\beta\widehat{H}_{\rm SS}$ incorporates negative-energy eigenvalues into the system. In fact, the structure of the first equation (15) reminds us again of the Zitterbewegung (ZB) phenomenon. On the other hand, note that, from the point of view of SUSYQM, $\widehat{\mathbf{Q}}$ and $\widehat{\mathbf{P}}$ are SUSY charges [4]:

$$\widehat{\mathbf{Q}}^2 = \frac{c}{3\hbar\omega^3}\widehat{H}_{\rm SS}, \qquad \widehat{\mathbf{P}}^2 = \frac{\hbar\omega}{3c^2}\widehat{H}_{\rm SS},$$

together with the second equation (15). Here,

$$\widehat{H}_{\rm SS} \equiv \frac{3}{2}\hbar\omega\left\{\left(\frac{c}{\hbar\omega}\right)^2\widehat{\mathbf{P}}^2 + \left(\frac{\omega}{c}\right)^2\widehat{\mathbf{Q}}^2\right\} = \frac{\widehat{\mathbf{p}}^2}{2m_0} + \frac{m_0\omega^2}{2}\widehat{\mathbf{q}}^2 + \frac{2\omega}{\hbar}\beta\mathbf{S}\cdot\mathbf{L}. \tag{16}$$

Furthermore,

$$[\widehat{H}_{\rm SS}, \widehat{\mathbf{Q}}] = [\widehat{\mathbf{Q}}^2, \widehat{\mathbf{Q}}] = \widehat{0}, \qquad [\widehat{H}_{\rm SS}, \widehat{\mathbf{P}}] = [\widehat{\mathbf{P}}^2, \widehat{\mathbf{P}}] = \widehat{0}, \tag{17}$$

which closes the algebra and confirms that $\widehat{H}_{\rm SS}$ is invariant under this supersymmetry. We note again that $H_{\rm SS} \geqslant 0$. That is, the Hamiltonian has only non-negative eigenvalues. Let us suppose that $|E_a\rangle$ is an eigenstate of $H_{\rm SS}$ with positive eigenvalue $E_a > 0$. Then it follows that $|E_a\rangle' \propto \widehat{Q}|E_a\rangle$ is also an eigenstate with the same positive eigenvalue. Relations (17), together with $\{\widehat{\mathbf{Q}}, \widehat{\mathbf{P}}\} = \widehat{0}$, are the graded algebra of a supersymmetric system consisting of a relativistic spin-1/2 particle interacting with an electric classical field [28, 29]. In the present case, the magnitude of the electric field is proportional to $r \equiv |\widehat{\mathbf{q}}| = |\mathbf{q}|$ (the gradient of $\mathbf{q}^2$).

3.2. Dynamical group symmetries

Some dynamical symmetries can be constructed for the harmonic oscillator in (3 + 1)-dimensions. To this end, let us assume that the quantum system is described by the vector-like $\widehat{q}^i, \widehat{p}^j$ $(i, j = 1, 2, 3)$ noncanonical 'internal' coordinates. We assume that they satisfy the commutation relations

$$[\widehat{q}^i, \widehat{p}^j] = \mathrm{i}\hbar\delta^{ij}\left(\widehat{I} + \frac{\chi}{\hbar\omega}\widehat{H}\right) \equiv \mathrm{i}\hbar\delta^{ij}\widehat{\mathcal{O}}, \tag{18}$$

in which $\widehat{H}$ is the Hamiltonian of the system, χ is a dimensionless real constant and ω is the frequency of the oscillator. For a harmonic oscillator we have to adjoin the commutation relations

$$\frac{\mathrm{d}\widehat{q}^j}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[\widehat{H}, \widehat{q}^j] = \frac{\widehat{p}^j}{m_0}, \qquad \frac{\mathrm{d}\widehat{p}^j}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[\widehat{H}, \widehat{p}^j] = -m_0\omega^2\widehat{q}^j. \tag{19}$$

Now we look for a dynamical symmetry (a Lie algebra) associated with this system. Jacobi's identity for the triad $\widehat{q}^i, \widehat{p}^i, \widehat{q}^j\,(i \neq j)$ is

$$[[\widehat{q}^i, \widehat{p}^i], \widehat{q}^j] + [[\widehat{q}^j, \widehat{q}^i], \widehat{p}^i] + [[\widehat{p}^i, \widehat{q}^j], \widehat{q}^i] = \widehat{0}, \tag{20}$$

with no summation on the i index. But from equation (18) $[[\widehat{p}^i, \widehat{q}^j], \widehat{q}^i] = \widehat{0}$. Then,

$$[[\widehat{q}^i, \widehat{q}^j], \widehat{p}^i] = [[\widehat{q}^i, \widehat{p}^i], \widehat{q}^j] = \mathrm{i}\hbar[\widehat{\mathcal{O}}, \widehat{q}^j]. \tag{21}$$

However, from equation (19), $[\widehat{\mathcal{O}}, \widehat{q}^j] \neq \widehat{0}$:

$$[\widehat{\mathcal{O}}, \widehat{q}^j] = \frac{\chi}{\hbar\omega}[\widehat{H}, \widehat{q}^j] = -\mathrm{i}\frac{1}{m_0}\frac{\chi}{\omega}\widehat{p}^j \neq \widehat{0} \tag{22}$$

only from the fact that $\chi \neq 0$. This means that $[\widehat{q}^i, \widehat{q}^j] \neq \widehat{0}$. Similarly, from the triad $\widehat{p}^i, \widehat{q}^i, \widehat{p}^j$ we find that $[\widehat{p}^i, \widehat{p}^j] \neq \widehat{0}$.

If we denote the generators of rotations for this system (i.e., the system's angular momentum components) by $\widehat{J}^k$, $k = 1, 2, 3$, which satisfy angular momentum commutation relations

$$[\widehat{J}^i, \widehat{J}^j] = \mathrm{i}\hbar\varepsilon^{ijk}\widehat{J}^k, \tag{23}$$

then

$$[\widehat{J}^i, \widehat{q}^j] = \mathrm{i}\hbar\varepsilon^{ijk}\widehat{q}^k, \qquad [\widehat{J}^i, \widehat{p}^j] = \mathrm{i}\hbar\varepsilon^{ijk}\widehat{p}^k. \tag{24}$$

Next, the commutators $[\widehat{q}^i, \widehat{q}^j]$ and $[\widehat{p}^i, \widehat{p}^j]$ transform as pseudo-tensors under space rotations. This can be seen by using Jacobi's identity for the triad $[[\widehat{q}^i, \widehat{q}^j], \widehat{J}^j]$ (no summation on j):

$$[[\widehat{q}^i, \widehat{q}^j], \widehat{J}^j] + [[\widehat{J}^j, \widehat{q}^i], \widehat{q}^j] + [[\widehat{q}^j, \widehat{J}^j], \widehat{q}^i] = \widehat{0}. \tag{25}$$

Because $[\widehat{q}^j, J^j] = \widehat{0}$, from equation (25) we get

$$[[\widehat{q}^i, \widehat{q}^j], J^j] + [[J^j, \widehat{q}^i], \widehat{q}^j] = \widehat{0}, \qquad [[\widehat{p}^i, \widehat{p}^j], J^j] + [[J^j, \widehat{p}^i], \widehat{p}^j] = \widehat{0}. \tag{26}$$

Thus, from equation (24) we obtain

$$[[\widehat{q}^i, \widehat{q}^j], J^j] = \mathrm{i}\hbar\varepsilon^{ijk}[\widehat{q}^i, \widehat{q}^j], \qquad [[\widehat{p}^i, \widehat{p}^j], J^j] = \mathrm{i}\hbar\varepsilon^{ijk}[\widehat{p}^i, \widehat{p}^j]. \tag{27}$$

The only pseudo-tensor available here is just J^k. Therefore, the solutions for the commutators $[\widehat{q}_i, \widehat{q}_j]$ and $[\widehat{p}_i, \widehat{p}_j]$, which are compatible with equations (24)–(27), become

$$[\widehat{q}^i, \widehat{q}^j] = \mathrm{i}\zeta\epsilon^{ijk}J^k, \qquad [\widehat{p}^i, \widehat{p}^j] = \mathrm{i}\xi\epsilon^{ijk}J^k, \tag{28}$$

where ζ and ξ are real constants (with clearly identifiable dimensions).

On the other hand, for the triad $\widehat{\mathcal{O}}, \widehat{q}^i, \widehat{q}^j\,(i \neq j)$ we have

$$[[\widehat{q}^i, \widehat{q}^j], \widehat{\mathcal{O}}] + [[\widehat{\mathcal{O}}, \widehat{q}^i], \widehat{q}^j] + [[\widehat{q}^j, \widehat{\mathcal{O}}], \widehat{q}^i] = \widehat{0}. \tag{29}$$

As $[\widehat{\mathcal{O}}, \widehat{q}^i] \propto \widehat{p}^i$, $[\widehat{q}^j, \widehat{\mathcal{O}}] \propto \widehat{p}^j$ and $[\widehat{q}^i, \widehat{p}^j] = \widehat{0}$ (for $i \neq j$), then $[[\widehat{q}^i, \widehat{q}^j], \widehat{\mathcal{O}}] = \widehat{0}$. Hence, given the fact that $[\widehat{q}^i, \widehat{q}^j] \propto \widehat{J}^k$, then

$$[\widehat{\mathcal{O}}, J^k] = [\widehat{H}, J^k] = \widehat{0}, \qquad k = 1, 2, 3. \tag{30}$$

This means that J^k are conserved quantities.

At this stage of the problem, one wonders what sort of relations there are among the various real parameters χ, ζ, ξ, etc that appear in the theory. To this end, let us choose the triad $\widehat{q}^l, \widehat{q}^m, \widehat{p}^i$. We then select equations (18), (19), (24), (28) to insert them (correspondingly) into Jacobi's identity

$$[[\widehat{q}^l, \widehat{q}^m], \widehat{p}^i] + [[\widehat{p}^i, \widehat{q}^l], \widehat{q}^m] + [[\widehat{q}^m, \widehat{p}^i], \widehat{q}^l] = \widehat{0}, \tag{31}$$

we obtain

$$\zeta\epsilon^{mlk}\epsilon^{kik'}\widehat{p}^{k'} - \hbar\chi\delta^{il}\widehat{p}^m + \hbar\chi\delta^{mi}\widehat{p}^l = \widehat{0}. \tag{32}$$

If $m = 1$, $l = 2$, and then making $i = 2$, we get $\zeta = -\chi/\omega$.

Following the same steps as above but for the triad $\widehat{p}^l, \widehat{p}^m, \widehat{q}^i$, we find that $\xi = -\chi m_0 \omega$. Therefore, if $\chi \gtrless 0$, then $\zeta \lessgtr 0$ and $\xi \lessgtr 0$. That is to say, $\xi = m_0 \omega^2 \zeta$.

Hence, the required commutation relations are

$$[\widehat{q}^l, \widehat{q}^m] = -\mathrm{i}\frac{\chi}{\omega}\epsilon^{lmk}\widehat{J}^k, \qquad [\widehat{p}^l, \widehat{p}^m] = -\mathrm{i}\frac{\chi\omega}{m_0}\epsilon^{lmk}\widehat{J}^k. \tag{33}$$

To sum up, the (order 10, rank 2) Lie algebra generated by $\widehat{q}^i, \widehat{p}^j, \widehat{J}^k$ and $\widehat{H}$ is given by the commutation relations

$$[\widehat{q}^i, \widehat{p}^j] = \mathrm{i}\hbar\delta^{ij}\left(\widehat{I} + \frac{\chi}{\hbar\omega}\widehat{H}\right), \tag{34}$$

$$[\widehat{H}, \widehat{q}^j] = -\mathrm{i}\hbar\widehat{p}^j, \qquad [\widehat{H}, \widehat{p}^j] = \mathrm{i}\hbar m_0\omega^2\widehat{q}^j, \tag{35}$$

$$[\widehat{q}^i, \widehat{q}^j] = -\mathrm{i}\frac{\chi}{\omega}\epsilon^{ijk}J^k, \qquad [\widehat{p}^i, \widehat{p}^j] = -\mathrm{i}\frac{\chi\omega}{m_0}\epsilon^{ijk}\widehat{J}^k, \tag{36}$$

$$[\widehat{J}^i, \widehat{J}^j] = \mathrm{i}\hbar\varepsilon^{ijk}\widehat{J}^k, \tag{37}$$

$$[\widehat{J}^i, \widehat{q}^j] = \mathrm{i}\hbar\varepsilon^{ijk}\widehat{q}^k, \qquad [\widehat{J}^i, \widehat{p}^j] = \mathrm{i}\hbar\varepsilon^{ijk}\widehat{p}^k, \tag{38}$$

$$[\widehat{H}, \widehat{J}^k] = \widehat{0}. \tag{39}$$

Formally, if $\chi \neq 0$, the central term $\mathrm{i}\hbar\delta^{ij}\widehat{I}$ in equation (34) can be reabsorbed into $\widehat{\mathcal{O}}$ and is mathematically unimportant for a finite-dimensional semisimple Lie algebra. However, we must distinguish the nature of the symmetry algebra according to the values that the real parameter 'χ' can have. There are three possible cases: (a) if $\chi \to 0$ and $\omega \to 0$ with $(\chi/\omega) \to \hbar/mc^2$, leading to the important case of the Poincaré algebra in (3 + 1)-dimensions. Additionally, the case $\chi \to 0$ leads to the (isotropic) harmonic-oscillator algebra in 3-dimensions, also known as the Newton–Hook algebra [30], the Heisenberg algebra being a subalgebra of it. Here $\mathrm{i}\hbar\delta^{ij}\widehat{I}$ is the non-trivial center of the algebra and it cannot be eliminated. (b) If $\chi < 0$, this symmetry corresponds to the compact Lie algebra $so(5)$. The generators $\widehat{q}^i, \widehat{p}^j, \widehat{J}^k$ and $\widehat{H}$ are traceless. This quantum model can be applied, for instance, to the study of the electron Zitterbewegung [3]. In this case then $\omega = \omega_{\mathrm{Zitt}} = 2m_0c^2/\hbar$. (c) If $\chi > 0$, the generated symmetry becomes the non-compact Lie algebra $so(3, 2)$. This algebra can be studied, for example, from the point of view of Dirac's representation [18]. This representation could be applied to the study of some hadron resonances [19] in the context of Regge trajectories. We recall here some early works which describe different systems of harmonic oscillators in terms of dynamical symmetry [20, 21]. Actually, there is an extensive (modern) literature on harmonic-oscillator models based on the dynamical algebra $so(3, 2)$ (see for instance [22–25]).

These Lie algebras are well known, particularly $so(5)$, so we are not going to get involved at present into the discussion of the (energy) spectrum of $\widehat{H}$ in terms of the eigenvalues of, say, J_3, etc. It is enough to say that it is possible to find closed results, according to the representation we use to describe the system in each case.

3.3. Canonical coordinates

Incidentally, we consider a general quantum system described by canonical coordinates $\widehat{Q}^\nu$ and $\widehat{P}^\mu$ satisfying the Heisenberg algebra [27]

$$[\widehat{P}^\mu, \widehat{Q}^\nu] = \mathrm{i}\hbar\mathbb{I}g^{\mu\nu}, \tag{40}$$

with the metric signature $g(+---)$, where $\mathbb{I} \equiv I_{n\times n} \otimes I$ represents an n-block identity matrix such that we may realize these operators in the general form

$$\widehat{Q}^{\nu} = \widehat{\eta}\widehat{q}^{\nu} \equiv \widehat{\eta} \otimes \widehat{q}^{\nu}, \qquad \widehat{P}^{\mu} = \widehat{\eta}\widehat{p}^{\mu} \equiv \widehat{\eta} \otimes \widehat{p}^{\mu}, \tag{41}$$

where, in coordinate representation, $\widehat{p}^{\mu} = -\mathrm{i}\hbar\partial/\partial q_{\mu}$. Here, $\widehat{\eta}$ is a constant $n \times n$ Hermitian matrix satisfying $\widehat{\eta}^2 = I_{n\times n}$. Thus we can define a label $|\mathrm{Tr}(\widehat{\eta})|$ associated with each representation of the Heisenberg algebra (40), with $n \geqslant |\mathrm{Tr}(\widehat{\eta})| \geqslant 0$. Representations satisfying $|\mathrm{Tr}(\widehat{\eta})| = n$ correspond to the usual ones ($\widehat{\eta} = I_{n\times n}$) where Q^{ν}, P^{μ} are reducible operators for $n \geqslant 2$.

The Hilbert space is defined as $L^2(\mathbb{R}^3) \otimes \mathbb{C}^n$. It consists of n-component column vectors where each component ψ_i is a complex-valued function of the (3 + 1)-dimensional (flat) spacetime coordinates $\mathbf{q}, t$. The coordinates $\widehat{Q}^i (i = 1, 2, 3)$ consist of three self-adjoint operators. The momentum operator $\widehat{P}^j = -\mathrm{i}\hbar\widehat{\eta}\partial/\partial q_j$ is defined as the Fourier transformation of the position operator $\widehat{Q}^j$.

Minimal interactions can now be introduced by means of the prescription $\widehat{P}^{\mu} \to \widehat{P}^{\mu} - g\widehat{A}^{\mu}$, where g is the coupling constant, A_{μ} is a gauge field ($\mu = 0, 1, 2, 3$). Note that here $\widehat{P}^0 = \mathrm{i}\hbar I_{n\times n}\partial/\partial q_0$. This is the basis of the so-called *gauge principle* whereby the form of the interaction is determined on the basis of local gauge invariance. The covariant derivative $D^{\mu} \equiv (\mathrm{i}/\hbar)(\widehat{P}^{\mu} - g\widehat{A}^{\mu})$ turns out to be of fundamental importance to determine the field strength tensor of the theory. It will be the operator which generalizes from electromagnetic-like interactions.

It is well known that the expression for the relativistic energy may be used to form the classical free-particle Hamiltonian. The analogous quantum mechanical expression could be constructed by replacing the classical momentum $\mathbf{p}$ with its quantum mechanical operator $\widehat{\mathbf{p}}$, with components $\widehat{p}^i = -\mathrm{i}\hbar\partial/\partial q_i$, which produces the spin-0 free-particle wave equation

$$\left(c^2\widehat{\mathbf{p}}^2 + m_0^2c^4\right)^{1/2}\Phi(q) = c\widehat{p}^0\Phi(q) = \mathrm{i}\hbar\frac{\partial}{\partial t}\Phi(q). \tag{42}$$

As is well known, this equation does not satisfy some of the conditions required by special relativity. The wave equation is not covariant, and the square root term introduces ambiguity. The Klein–Gordon equation (KGE) solves both of these problems simply by taking the square of the original energy expression and extending the result to a quantum mechanical wave equation:

$$\left(g_{\mu\nu}\widehat{p}^{\mu}\widehat{p}^{\nu} - m_0^2c^2\right)\Phi(q) = 0, \tag{43}$$

where $\widehat{p}^{\mu} = -\mathrm{i}\hbar\partial/\partial q_{\mu}$ [26]. The resulting wave equation is covariant, but suffers from other problems. Negative energy solutions to this equation are possible, which do not have a readily obvious explanation. Besides, the probability density $\Phi^*\Phi$ fluctuates with time.

The expression in equation (43) is still valid if we make the replacement $\widehat{p}^{\mu} = -\mathrm{i}\hbar\partial/\partial q_{\mu} \to \widehat{P}^{\mu} = \sigma_1\widehat{p}^{\mu}$, with $\sigma_{j=1}$ a Pauli matrix, so that

$$\left(m_0^2c^4 - \widehat{P}^j\widehat{P}_j\right)\Phi(q) = (\widehat{P}^0)^2\Phi(q) = \left(m_0^2c^4 - \widehat{p}^j\widehat{p}_j\right)\Phi(q) = (\widehat{p}^0)^2\Phi(q), \tag{44}$$

since $[\widehat{p}^i, \sigma_j] = \widehat{0}$ and $\sigma_j^2 = I_{2\times 2}$. Note that Φ is now a two-component wavefunction, that is to say, the Hilbert space has been enlarged. Pauli matrices σ_j satisfy the well-known relations

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}, \qquad \sigma_j^2 = I_{2\times 2}, \qquad i, j, k = 1, 2, 3, \qquad \sigma_i\sigma_j = i\sigma_k \text{ cyclically}. \tag{45}$$

Next we briefly discuss the interaction of a basic effective minimal coupling between spinless quarks [31]. Let

$$\widehat{P}^0 \to \widehat{\Pi}^0 = \mathrm{i}\hbar c\sigma_1\nabla^0 - \left(\sigma_2\sqrt{\frac{-a_1^2}{r}} + a_2\sigma_3\sqrt{r}\right), \tag{46}$$

$$\widehat{P}^k \rightarrow \widehat{\Pi}^k = \mathrm{i}\hbar\sigma_1 \nabla^k - a_3\sigma_2 \frac{q^k}{r} \tag{47}$$

be a minimal replacement in the KGE, with a_1, a_2, a_3 real constants. If we introduce the minimal coupling $\widehat{P}^\mu \rightarrow \widehat{\Pi}^\mu$ into equation (44) and divide the resulting equation by $2m_0c^2$, we get

$$\left\{-\frac{\hbar^2}{2m_0}\nabla^2 - \left(\frac{a_1^2}{2m_0c^2} - 2\hbar a_3\sigma_3\right)\frac{1}{r} + \frac{a_2^2}{2m_0c^2}r + \frac{1}{2}m_0c^2\left(1 + \left(\frac{a_3}{m_0c^2}\right)^2\right)\right\}\Phi(q) = -\frac{\hbar^2}{2m_0}\frac{\partial^2}{\partial t^2}\Phi(q). \tag{48}$$

Note that $\widehat{\Pi}^0$ is not Hermitian. However when placed into the KGE, each piece becomes in fact Hermitian. This feature also takes place in the Dirac oscillator definition [12]. Rearranging terms in equation (48) we obtain the relativistic wave equation

$$\left(-\frac{\hbar^2}{2m_0}\nabla^2 - \frac{\alpha}{r} + kr\right)\Phi(\mathbf{q}) = \epsilon_{\mathrm{KG}}\Phi(\mathbf{q}), \tag{49}$$

with

$$\epsilon_{\mathrm{KG}} \equiv \frac{1}{2}m_0c^2\left(\left(\frac{E_{\mathrm{KG}}}{m_0c^2}\right)^2 - \left(\frac{a_3}{m_0c^2}\right)^2 - 1\right), \qquad \alpha \equiv \frac{a_1^2}{2m_0c^2} - 2\hbar a_3\sigma_3, \qquad k \equiv \frac{a_2^2}{2m_0c^2}. \tag{50}$$

Of course, in this example we are not dealing with electromagnetic-like interactions. Note that equation (48) has been obtained (as a minimal coupling) within the framework of a relativistic equation. Furthermore, for the free particle ($a_1 = a_2 = a_3 = 0$) the limit $|\widehat{\mathbf{p}}| \rightarrow 0$ yields

$$\epsilon_{\mathrm{KG}} = \frac{1}{2}m_0c^2\left(\left(\frac{E_{\mathrm{KG}}}{m_0c^2}\right)^2 - 1\right) \rightarrow \frac{1}{2}m_0c^2\left(\left(\frac{\left(c^2\widehat{\mathbf{p}}^2 + m_0^2c^4\right)^{1/2}}{m_0c^2}\right)^2 - 1\right) \rightarrow \frac{\widehat{\mathbf{p}}^2}{2m_0}, \tag{51}$$

as one should anticipate. Thus, we reacquire the nonrelativistic expression for the kinetic energy.

We also have that

$$F^{0k} \propto [\widehat{\Pi}^0, \widehat{\Pi}^k] = -\mathrm{i}\hbar\left(a_2\sigma_2 + \frac{1}{2}\frac{a_1}{r^{3/2}}\sigma_3\right)\widehat{q}^k, \tag{52}$$

$$F^{ij} \propto [\widehat{\Pi}^i, \widehat{\Pi}^j] = -\mathrm{i}a_3\frac{\sigma_3}{r}L_k, \qquad i, j, k \text{ cyclically}. \tag{53}$$

Note as well that in this case

$$[A^0, A^k] = -2\mathrm{i}a_2a_3\frac{q^k}{\sqrt{r}}\sigma_1 \neq \widehat{0}, \tag{54}$$

so that the vector potential A^μ is not Abelian. In QCD, the strong color field is mediated by massless vector bosons. Hence, the potential might be expected to be of the Coulomb form. At large r, the quarks are subjected to confining forces. It is found that the potential at large r is linear [31]. We are not going to solve here the eigenvalue problem (49), since it has been widely studied in the context of the (nonrelativistic) Schrödinger equation [2]. Note that this basic model by no means solves the general problem of, for instance, heavy quarkonia in the relativistic quark model [32].

4. Conclusions

This paper deals with some possible applications of particular harmonic-oscillator-like models to particle physics in the framework of certain noncanonical commutation relations. We have presented specific types of harmonic oscillators in order to study mathematical relations among well-established symmetries in physics such as supersymmetry, quark interactions and particle–antiparticle symmetries. It remains to be given a more solid foundation for this portrayal. For instance, if one wants to be meaningful, there is a need (a difficult one) to incorporate various degrees of freedom existing in meson–quark physics, accounting for physical effects such as retardation and radiative corrections, amongst others [32]. Perhaps one way of particularly doing this is by formally generalizing equation (1) as

$$[\widehat{q}^i, \widehat{p}^j] = \mathrm{i}\hbar\delta^{ij}(\widehat{I} + f(\widehat{O}_1, \ldots, \widehat{O}_n)), \tag{55}$$

where $f(\widehat{O}_1, \ldots, \widehat{O}_n)$ is a power series function of the constant n-compatible observables $\widehat{O}_j$ of the system, i.e., they commute with the Hamiltonian and with each other. Actually, at present our research is directed to extract information from equation (55) for simple systems in $(3 + 1)$-dimensions, where a somehow judicious form for the function f is to be set.

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